Generalised q-Deformed Oscillators and their Statistics

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Abstract

We consider a version of generalised q-oscillators and some of their applications. The generalisation includes also "quons" of infinite statistics and deformed oscillators of parastatistics. The statistical distributions for different q-oscillators are derived for their corresponding Fock space representations. The deformed Virasoro algebra and SU(2) algebra are also treated.

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1 Introduction

Over the last few years there has been a growing activity in the study of Quantum Groups and Algebras [1]-[3]. This is connected with the fact that these new mathematical structures are relevant for a variety of diverse problems in Theoretical Physics, such as quantum inverse scattering theory, exactly solvable model in statistical mechanics, rational conformal field theory, two-dimensional field theory with fractional statistics, etc...

The algebraic structure of quantum group can be formally described as a q-deformation of the enveloping algebra $U(G)$ of a Lie algebra G in such a way that G is recovered in the limit of the deformation parameter $q \to 1$. In this connection the concept of q-deformed quantum harmonic oscillators has also been introduced [4, 5], which prove to be powerful for studying the representation of q-deformed enveloping algebra. In particular the q-formalism has been developed for $SU(2)$ group and some others.

The study of q-oscillators has been, on the other hand, stimulated during the recent years by the increasing interests in particles obeying statistics different from Bose and Fermi.

In this paper we would like to consider a version of generalised q-oscillators and some of their applications. This generalisation includes on an equal footing the usual q-deformed oscillators [4, 5] and the "quons" of infinite statistics [6]-[9]. We also extend the formalism for the deformation of parastatistics [10, 11]. The contents of the paper are arranged as follows. In section 2 we consider the q-deformed single mode oscillators, their satatistics, and associated q-deformed Virasoro algebra. In section 3 we are dealing with multimode oscillators and the representations of (q-deformed) $SU(2)$ algebra. Section 4 is devoted to q-deformed paraoscillators.

2 Single Mode Oscillator.

1. We consider the bosonic oscillator with the deformed commutation rule of the form:

$$aa^+ - qa^+a = q^{cN} \tag{2.1}$$

where N is oscillator number operator,

$$[N, a] = -a \tag{2.2}$$

q and c are some parameters.

The usual q-deformation

$$aa^+ - qa^+a = q^{-N}$$

corresponds to the value $c = -1$, and the "infinite statistics"

$$aa^+ = 1 \tag{2.3}$$

corresponds to $c = 0, q = 0$.

Equation (2.1) gives:

$$aa^{+n} = q^n a^{+n} a + [n]_q^{(c)} (a^+)^{n-1} q^{cN} \tag{2.4}$$

where the general notation

$$[x]_q^{(c)} \equiv \frac{q^x - q^{cx}}{q - q^c} \tag{2.5}$$

is used.

It is seen from the equation (2.4) that the algebra (2.1) can be realised in the Fock space spanned by the orthonormalised eigenstates of the operator N,

$$|n> \equiv \frac{1}{\sqrt{[n]_q^{(c)}!}} (a^+)^n |0> \tag{2.6}$$

and in this space the following relations hold:

$$a^+ a = [N]_q^{(c)}, \quad aa^+ = [N+1]_q^{(c)} \tag{2.7}$$

For the calculations it is helpful to use the identities:

$$\begin{aligned}
[x]_q^{(c)} &= [x]_{q^c}^{(c^{-1})} \\
[-x]_q^{(c)} &= -q^{-(c+1)} [x]_{q^{-1}}^{(c)} \\
[x+y]_q^{(c)} &= q^y [x]_q^{(c)} + q^{cx} [y]_q^{(c)} = q^{cy} [x]_q^{(c)} + q^x [y]_q^{(c)}
\end{aligned} \tag{2.8}$$

2. For deformed fermionic oscillator we put:

$$bb^+ + q b^+ b = q^{cN}, \quad [N, b] = -b \tag{2.9}$$

This algebra can be realised in the Fock space spanned by the orthonormalised eigenstates of N,

$$|n> = \frac{1}{\sqrt{\{n\}_q^{(c)}!}} \, b^{+n} |0> \tag{2.10}$$

Here the notation

$$\{n\}_q^{(c)} = \frac{q^{cn} + (-1)^{n+1} q^n}{q^c + q} \tag{2.11}$$

is used.

Using the equation

$$bb^{+n} = (-1)^n q^n b^{+n} b + \{n\}_q^{(c)} (b^+)^{n-1} q^{cN} \tag{2.12}$$

it can be shown that in this Fock space the following relations hold:

$$b^+ b = \{N\}_q^{(c)}, \; bb^+ = \{N+1\}_q^{(c)} \tag{2.13}$$

3. Consider now the deformed Green function defined as the statistical distribution of a^+a and b^+b. The statistical distribution of the operator F is defined through the formula:

$$< F >= \frac{1}{Z}\, Tr\left(e^{-\beta H} F\right) \tag{2.14}$$

where Z is the partition function,

$$Z \equiv Tr(e^{-\beta H})$$

which determines the thermodynamic properties of the system, $\beta = \frac{1}{KT}, H$ is Hamiltonian, which is usually taken of the form $H = \omega N$, ω being one particle-oscillator energy. The trace must be taken over a complete set of states.

The calculations give the following results:

$$< a^+a > \;=\; \frac{e^{\beta\omega} - 1}{e^{2\beta\omega} - (q + q^c)e^{\beta\omega} + q^{1+c}} \tag{2.15}$$

$$< b^+b > \;=\; \frac{e^{\beta\omega} - 1}{e^{2\beta\omega} + (q - q^c)e^{\beta\omega} - q^{1+c}} \tag{2.16}$$

From these we recover the familiar formulae

$$< a^+a >= \frac{1}{e^{\beta\omega} - 1}$$

$$< b^+b >= \frac{1}{e^{\beta\omega} + 1}$$

for Bose and Fermi statistics when $q = 1$, and the result [12]

$$< a^+a > \;=\; \frac{e^{\beta\omega} - 1}{e^{2\beta\omega} - (q + q^{-1})e^{\beta\omega} + 1}$$

$$< b^+b > \;=\; \frac{e^{\beta\omega} - 1}{e^{2\beta\omega} + (q - q^{-1})e^{\beta\omega} + 1}$$

for usual q-deformed statistics when $c = -1$.

In the limit of "infinite statistics", $c = 0, q = 0$, we have:

$$< a^+a >=< b^+b >= e^{-\beta\omega} \tag{2.17}$$

4. The Virasoro algebra plays a crucial role in string theory. The centreless Virasoro algebra consists of the generators $L_n, n \in Z$, satisfying the commutation relation:

$$[L_n, L_m] = (n - m)L_{n+m} \tag{2.18}$$

In the (undeformed) oscillator formalism this algebra can be realised by putting

$$L_n = (a^+)^{-n+1} a \tag{2.19}$$

Consider now the q-deformation of Virasoro algebra based on the q-oscillator algebra (2.1). A version of such deformation can be obtained if instead of (2.19) we take the generators of the form:

$$\widetilde{L}_n = q^{cN}(a^+)^{-n+1}a \tag{2.20}$$

We then have:

$$\widetilde{L}_n\widetilde{L}_m - q^{(c+1)(n-m)} \;\; \widetilde{L}_m\widetilde{L}_n = q^{c(2N+n+m)}[n-m]_q^{(c)} \;\; \widetilde{L}_{n+m} \tag{2.21}$$

The usual q-deformed Virasoro algebra

$$[\widetilde{L}_n, \widetilde{L}_m] = q^{-(2N+n+m)}[n-m]_q \;\; \widetilde{L}_{n+m}$$

is recovered by putting $c = -1$.

For the case of infinite statistics, $c = 0, q = 0$, equation (2.21) gives:

$$\widetilde{L}_n\widetilde{L}_m = \widetilde{L}_{n+m} \tag{2.22}$$

This result can also be checked directly from (2.3) and the expression of $\widetilde{L}_n$,

$$\widetilde{L}_n = (a^+)^{-n+1}a$$

followed from (2.20) with $c = 0, q = 0$.

3 Multimode Oscillator.

The formulae (2.1) and (2.9) for single mode oscillators can be generalised for multimode oscillators as follows:

$$A_iA_j^+ \mp \left\{(q - q^{\delta(c)})\delta_{ij} + q^{\delta(c)}\right\} A_j^+A_i = \delta_{ij}\; q^{cN_i} \tag{3.1}$$

$$e^{cA_i}A_j\; e^{-cA_i} = A_j \tag{3.2}$$

$$[N_i, A_j] = -\delta_{ij}\; A_i \tag{3.3}$$

where A stands for a or b,

$$\delta(c) = \begin{cases} 1 & , \quad c = 0 \\ 0 & , \quad c \neq 0 \end{cases}$$

and the upper (lower) sign in (3.1) refers to the bosonic (fermionic) case.

Equation (3.2) means that $[A_i, A_j] = 0$ when $c \neq 0$, but no commutation rule is imposed on A_iA_j when $c = 0$. In the latter case (3.1) reads [12]:

$$A_iA_j^+ \mp q\; A_j^+A_i = \delta_{ij} \tag{3.4}$$

Let us consider now the realisation of (deformed) $SU(2)$ algebra based on the q-oscillator algebra (3.1)-(3.3). In the case $c \neq 0$ this can be performed in the Fock space spanned by the orthonormalised eigenstates of N_1 and N_2 defined as

$$|jm>= \frac{1}{\sqrt{[j-m]_q^{(c)}![j+m]_q^{(c)}!}}\ (a_1^+)^{j+m}(a_2^+)^{j-m}|0> \qquad (3.5)$$

with the identification:

$$\begin{aligned} E &= q^{-\frac{1}{2}(1+c)N_2} {a_1}^+ a_2 \\ F &= q^{-\frac{1}{2}(1+c)(N_2-1)} {a_2}^+ a_1 \\ H &= N_1 - N_2 \end{aligned} \qquad (3.6)$$

In fact, using the identities (2.8) it can be shown that

$$\begin{aligned} &[H,E]=2E, \quad [H,F]=-2F \\ &EF - q^{-(1+c)} \quad FE = [H]_q^{(c)} \end{aligned} \qquad (3.7)$$

From here, in particular, we recover the usual one-parameter deformed $SU(2)_q$ algebra [4, 5] when $c=-1$:

$$[E,F] = [H]_q \equiv \frac{q^H - q^{-H}}{q-q^{-1}}$$

In the case of infinite statistics, $c=0, q=0$, the representation of $SU(2)$ can be constructed in the Fock space spanned by the orthonormalised eigenstates

$$|jm;(r),(s)>= \prod_{K=1}^{r} (a_1^+)^{r_K}(a_2^+)^{S_K}|0> \qquad (3.8)$$

$$\sum_{K=1}^{p} r_K = j+m, \qquad \sum_{K=1}^{p} S_K = j-m$$

by putting

$$\begin{aligned} E &= \sum_{i=1,2} \sum_{K=0}^{\infty} a_{i_K}^+ ... a_{i_2}^+ a_{i_1}^+ . a_1^+ a_2 . a_{i_1} . a_{i_2} ... a_{i_K} \\ F &= \sum_{i=1,2} \sum_{K=0}^{\infty} a_{i_K}^+ ... a_{i_2}^+ a_{i_1}^+ . a_2^+ a_1 . a_{i_1} . a_{i_2} ... a_{i_K} \\ H &= N_1 - N_2 \\ N_j &= \sum_{i=1,2} \sum_{K=0}^{\infty} a_{i_K}^+ ... a_{i_2}^+ a_{i_1}^+ . a_j^+ a_j . a_{i_1} . a_{i_2} ... a_{i_K} \end{aligned} \qquad (3.9)$$

Note that the dimension of representation with spin j is $2j+1$ for $j = 0, \frac{1}{2}$, and more than $2j+1$ for $j \geq 1$. Thus, we have two-dimensional representation for $j = \frac{1}{2}$ with state vectors:

$$|\frac{1}{2}, \frac{1}{2} > = a_1^+|0>, \qquad |\frac{1}{2}, -\frac{1}{2} >= a_2^+|0>,$$

four-dimensional representation for $j = 1$ with state vectors:

$$|1,1>= (a_1^+)^2|0>$$

$$|1,0;(1)>= a_1^+a_2^+|0>, \quad |1,0;(2)>= a_2^+a_1^+|0>$$

$$|1,-1>= (a_2^+)^2|0>$$

etc.

4 Deformed para-Oscillator.

Here we restrict the consideration to the case of single mode parabose oscillator only. As has been shown in [13, 14] the (undeformed) parabose oscillator of order p obeys the commutation relations:

$$\begin{aligned} [a, a^+] &= 1 + (-1)^N(p-1) \\ [N, a] &= -a \\ N &= \frac{1}{2}(a^+a + aa^+ - p) \end{aligned} \tag{4.1}$$

In the Fock space spanned by the state vectors $|n> \sim (a^+)^n|0>$, (4.1) gives:

$$\begin{aligned} aa^+|n> &= |n>.\begin{cases} (n+p), & n \text{ even} \\ (n+1), & n \text{ odd} \end{cases} \\ a^+a|n> &= |n>.\begin{cases} n, & n \text{ even} \\ (n+p-1), & n \text{ odd} \end{cases} \end{aligned} \tag{4.2}$$

For the q-deformation we propose:

$$\begin{aligned} aa^+|n> &= |n>.\begin{cases} [n+p]_q^{(c)}, & n \text{ even} \\ [n+1]_q^{(c)}, & n \text{ odd} \end{cases} \\ a^+a|n> &= |n>.\begin{cases} [n]_q^{(c)}, & n \text{ even} \\ [n+p-1]_q^{(c)}, & n \text{ odd} \end{cases} \end{aligned} \tag{4.3}$$

as the generalisation of (4.2).

This means that in the Fock space of states $|n>$ the following equations hold:

$$\begin{aligned} aa^+ &= \frac{1}{2}(1+(-1)^N)[N+p]_q^{(c)} + \frac{1}{2}(1-(-1)^N)[N+1]_q^{(c)} \\ a^+a &= \frac{1}{2}(1+(-1)^N)[N]_q^{(c)} + \frac{1}{2}(1-(-1)^N)[N+p-1]_q^{(c)} \end{aligned} \tag{4.4}$$

From these, by using the identities (2.8), we can derive the following commutation relation:

$$\begin{aligned} & aa^+ - q^{1+(p-1)(-1)^N} a^+a \\ &= \left\{ \frac{1}{2}(1+(-1)^N)[p]_q^{(c)} - \frac{1}{2}(1-(-1)^N)[p-2]_q^{(c)}.q^{c+2-p} \right\} q^{cN} \end{aligned} \tag{4.5}$$

From (4.5) we can obtain the recurrent formula:

$$\begin{aligned} aa^{+n} &= q^{n+\frac{1}{2}(1-(-1)^n).(p-1)(-1)^N} a^{+^n} a \\ &+ \frac{1}{2} \sum_{K=0}^{[\frac{n-1}{2}]} q^{2(1-c)K} f(N)\ q^{cN} (a^+)^{n-1} \qquad (4.6) \\ &+ \frac{1}{2} \left(\sum_{K=0}^{[\frac{n}{2}]} q^{2(1-c)K} - 1 \right) q^{c-1+(p-1).(-1)^N} f(N-1) q^{cN} (a^+)^{n-1} \qquad (4.7) \end{aligned}$$

where

$$f(N) \equiv (1+(-1)^N)[p]_q^{(c)} - (1-(-1)^N)[p-2]_q^{(c)}.q^{c+2-p}$$

Equation (4.6) allows us to have the following expression for the norm of the state $(a^+)^n|0>$:

$$\begin{aligned} & <0|a^n a^{+n}|0> \\ &= \frac{1}{2}\ q^{c(n-1)} \left\{ \sum_{K=0}^{[\frac{n-1}{2}]} q^{2(1-c)K} f(n-1) + \left(\sum_{K=0}^{[\frac{n}{2}]} q^{2(1-c)K} - 1 \right) .\ q^{c-1-(p-1)(-1)^n} f(n) \right\} \end{aligned} \tag{4.8}$$

Finally, let us be interested in the statistical distribution of a^+a. The calculations give the following result:

$$<a^+a>= \frac{1-e^{-\beta\omega}}{q-q^c} \left\{ \frac{1+q^p e^{-\beta\omega}}{1-q^2 e^{-2\beta\omega}} - \frac{1+q^{cp}e^{-\beta\omega}}{1-q^{2c}e^{-2\beta\omega}} \right\} \tag{4.9}$$

It is straighforward to check that when $p=1$ this recovers the formula (2.15).

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